Publ. Astron. Obs. Belgrade No. 80 (2006), 173 - 177 *Contributed paper*

FULLY ANALYTICAL KINETIC MODEL OF RESONANCE DYNAMICS IN THE SOLAR SYSTEM

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Abstract. We propose a fractional kinetic equation to model the transport in eccentricity of objects in the mean motion resonances in the Elliptic Planar Restricted Three-Body Problem. Making use of the renormalization group formalism, we show how the fractional exponents and the diffusion coefficient can be estimated analytically, making use of the degeneracy of the problem. We apply our model to selected Mean Motion Resonances in the Solar System and explain some basic properties of transport in these resonances.

1. INTRODUCTION

Kinetic models of chaotic transport in the Asteroid Belt have been proposed or discussed by a number of authors (Varvoglis and Anastasiadis, 1996; Murray and Holman, 1997; Tsiganis, Varvoglis and Hadjidemetriou, 2002). All of these models are based on a Fokker-Planck type equation ("normal" diffusion equation) for the eccentricity, which implicitly assumes the Gauss-Markovian statistics of the quasi-random walk in the eccentricity space. In order to incorporate the more general (and more realistic) case of Levy-type statistics (see, e.g., Zaslavsky, 2002), we have recently proposed a semi-analytical model (Čubrović, 2005a) based on the fractional kinetic equation – FKE (Zaslavsky, 2002). In this paper we give a fully analytical model.

We consider the transport in Mean Motion Resonances (MMR) of the Elliptic Planar Restricted Three-Body Problem (EPRTBP), and apply it on diffusion in the asteroid belt. The model we propose is, however, applicable to any EPRTBP Hamiltonian, and possibly also to a wider class of degenerate systems, which all have a similar structure of resonances.

2. THE MODEL OF TRANSPORT

Our model is the Hamiltonian for a q-th order two-body MMR, used, among others, by Murray and Holman (1997):

$$H_{MMR}(L, \lambda; P, p) = H_0(L) + \sum_s c_s(L) P^s \cos \sigma_s \tag{1}$$

with L, λ, $P = L(1 - \sqrt{1 - e^2})$, and $p = -\widetilde{\omega}$ being the modified Delauney variables (defined as in Morbidelli 2002). In the above relations, e is eccentricty, and ω is the longitude of perihelion. Critical angles are denoted by σ_s. We assume that the transport only takes place along P (i.e., that the timescale of transport in L is much longer) and that a particle starts at $P = 0$, where we put a reflecting barrier, and escapes immediately after reaching a planet-crossing orbit ($P = P_{\rm cross}$), where we put an absorbing barrier. The possible phase protection mechanisms are ignored.

Assuming a Levy-type statistics for the angles σ_s, one finds, after performing the averaging, that the master equation for the probability distribution function $f(P, t)$ leads to a multi-channel FKE:

$$\frac{\partial^\beta f(P, t)}{\partial t^\beta} = \sum_{s=1}^{q} \frac{\partial^{\alpha s}}{\partial |P|^{\alpha s}} \left[D_s(P) f(P, t) \right] \tag{2}$$

with the diffusion coefficients:

$$D_s = \frac{1}{2} c_s^{\alpha s} s^{\alpha s} P^{s^2 \alpha} T_{\rm Lib}^{(\alpha - \beta) s} \tag{3}$$

where $T_{\rm Lib}$ denotes the libration period. Applying the separation ansatz and writing the time-independent part of the solution as a superposition of the time-independent parts of one-channel solutions (given in Čubrović, 2005b) for different values of s (which is justified by the linearity of FKE), one finds an estimate for the removal time and the Lyapunov time (for the latter from the FKE for the variational equations, which shares all the basic properties with (2)):

$$T_R \approx \left(\frac{P_0 P_{\rm cross}}{D(L, P_0) D(L, P_{\rm cross})} \right)^{1/\beta} \times \Phi \left(\alpha, \beta; L, P_0; \cos \left(\frac{\log P_0}{|\log \beta|} \right) \right) \tag{4}$$

$$T_{\rm Ly} \approx 2 \frac{c_s(L)}{\|D\|} \tag{5}$$

where $\|\|$ denotes the standard Euclidean norm, and Φ is, in general, a complicated resonance-dependent function which, however, has an important property of log-periodicity.

A few interesting consequences follow from the above results. First, it is obvious that, for different values of P, different components D_s of the diffusion coefficient will prevail, leading to a stair-like behavior of the "effective" (e.g., numerically computed) diffusion coefficient. Furthermore, since the amplitude of log-periodic oscillations of Φ can be shown to grow with q (order of the resonance), the log-periodic oscillations will be more and more significant for higher order resonances. Also, applying the Tauberian theorem for the Fourier transform, and making use of the generalized Central Limit Theorem (e.g., Weiss, 1994), one can show that two *approximate* scalings of $T_{\rm R}$ with $T_{\rm Ly}$ are possible: the power-law one and the stretched-exponential one (corresponding to the "stable chaos").

In order to actually compute any of the relevant quantities (like $T_{\rm R}$), however, the question of determining the exponents α and β arises. We will sketch in the next section an analytical procedure to do that.

3. THE RENORMALIZATION GROUP EQUATION

The idea of the Renormalization Group of Kinetics (RGK) is to model the transport explicitly as a random walk in P with the waiting time distribution $\Psi(t)$ and the step-size distribution $W(\Delta P)$ being chosen from the dynamical considerations (see Kuznetsov and Zaslavsky, 1997). Although also the FKE (2) could have been deduced from this formalism, we have decided to retain the more common averaging procedure for obtaining the FKE.

Two basic mechanisms of transport are expected to be the "hopping" between subsequent layers in the resonant multiplet, and the trapping inside higher and higher levels of hierarchy of a cantorus or a stability island chain. This process leads to the following expressions for $\Psi(t)$ and $W(\Delta P)$:

$$\Psi(t) = \text{const.} \times \sum_{j=1}^{N} p^j \left[\exp(-b^j t/T_{\rm Lib}) + \exp(-t/jT_{\rm Lib})\right] \tag{6}$$

$$W(\Delta P) = \text{const.} \times \sum_{j=1}^{N} p^j \left[\delta(\Delta P + a_0 a^j) + \delta(\Delta P - a_0 a^j) + \delta(\Delta P + j a_0) + \delta(\Delta P - j a_0)\right] \tag{7}$$

where $\delta(x)$ is the common Dirac delta function. The above functions obey the following RGK in the Fourier-Laplace $(q,\, u)$ space:

$$W(q) \to pW(qa), \quad \Psi(u) \to p\Psi(u/b), \quad j \to 2j, \quad N \to N/2 \tag{8}$$

which leads to two coupled fixed-point equations for a and b, whereas $p = \delta S/\Delta S$ (the relative overlap, which can be estimated, e.g., as in Murray and Holman, 1997), and a_0 is also easy to calculate as the separation between the subsequent resonant layers. Now the fractional exponents are found as $\alpha = |\log p|/\log a$ and $\beta = |\log p|/\log b$. Notice that the presented model breaks down in a non-degenerate system, where $a_0 \to 0$ and we only have a trivial RGK with a whole interval of fixed points (irrelevant for our purposes).

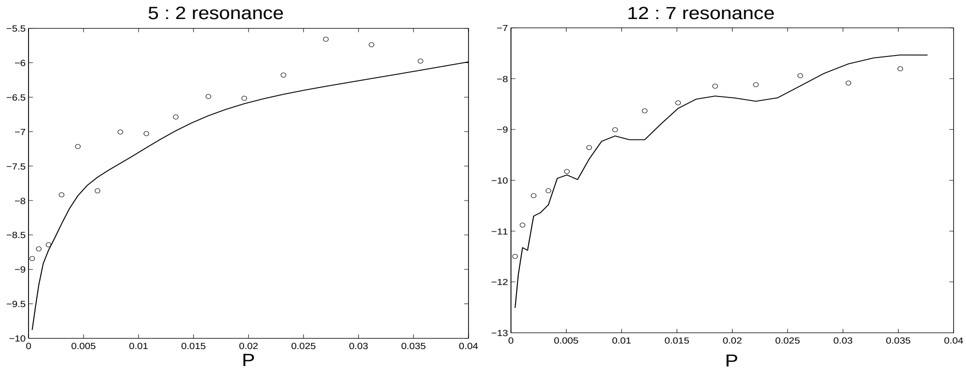

Figure 1: Diffusion coefficient as a function of P in the 5 : 2 (left) and 12 : 7 (right) resonances. Solid line is the analytical prediction, and circles are numerical results for an ensemble of 150 bodies. The value of the diffusion coefficient varies rather smoothly with P in the former case, whereas in the latter case the stair-like structure and the log-periodic oscillations are much more prominent.

4. RESULTS FOR SELECTED RESONANCES IN THE ASTEROID BELT

A systematic study of all important MMR in the Asteroid Belt is still in progress. We present here only two typical cases, namely the 5 : 2 and 12 : 7 resonances. Figure 1 shows the diffusion coefficient as a function of P; figure 2 gives the predicted T_{R}–T_{Ly} relation.

The figures illustrate the qualitative properties discussed at the end of the second section. The 12 : 7 resonance, as expected, contains a population with very long removal times, in which the bodies such as the now famous 511 Helga (Milani and Nobili, 1992) reside. The power-law and the exponential-law regimes for T_{R} coexist in a large part of the resonance. Contrary, the 5 : 2 objects should all have comparable lifetimes, and only one type of transport should exist – the relatively fast diffusion towards P_{cross}.

5. CONCLUSIONS

We have proposed a fractional kinetic equation for the eccentricity transport in the MMR, together with an analytical scheme for the estimation of the fractional exponents α and β. The predictions for two typical low- and high-order resonances agree well with the numerical results and with our previous, semi-analytical model (Tsiganis, Anastasiadis and Varvoglis, 2000; Čubrović, 2005a). It is clear, however, that further work is needed to obtain a complete model, capable of describing also the fine details of dynamics inside the MMR.

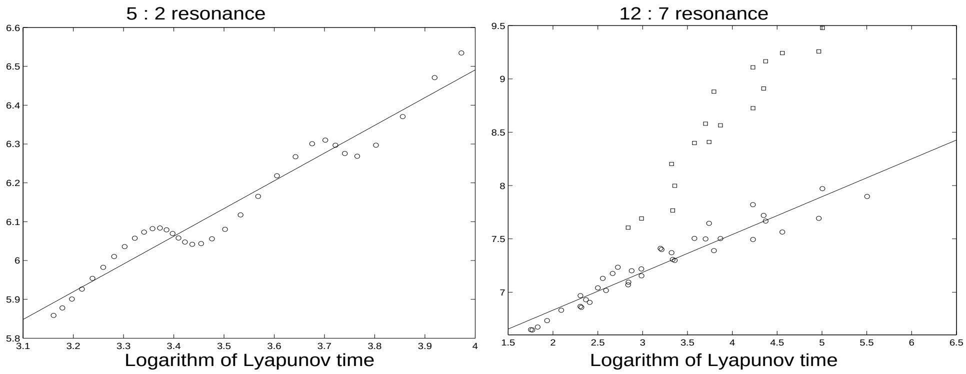

Figure 2: The analytically computed values of T_{Ly} and T_{R} for the 5 : 2 (left) and 12 : 7 (right) resonances. The power law (straight line on the log-log plot) describes well the general trend in the 5 : 2 case, with the scaling exponent about 0.70. In the 12 : 7 case, however, another, more stable regime, with exponentially long lifetimes is also present (denoted by squares), in addition to the "normal" chaos, which is again well described with a power-law fit, the slope being about 0.35.

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